The Pedagogy of Mathematical Inquiry

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Most problems given to children in school mathematics are clearly stated, take only a few minutes to answer, include little or no use of context and have a single correct answer. In mathematical inquiry, often problems are riddled with ambiguity, can require days or even weeks to address, necessitate an indepth understanding of the surrounding contextual issues and have no single correct answer. Because of this, the pedagogical practices that underpin an inquiry-based approach in mathematics are quite different than those in more conventional classrooms. This chapter explains the key elements that distinguish mathematical inquiry from more conventional mathematical tasks and what this means for teachers. Four phases of inquiry (Discover, Devise, Develop, Defend) are used to outline the complexity and diversity of practices that teachers need to adapt into their pedagogical practices. To illustrate inquiry-based teaching in practice, a case study is used from a teacher's classroom of Year 6/7 students (ages 10-13). The case study comes from a longitudinal study aimed at understanding the evolving pedagogies and experiences of primary teachers as they adapted their teaching practices to incorporate mathematical inquiry. In the case study, two consecutive units with similar mathematical structures and learning goals are provided along with interviews from the teacher following each unit to highlight many of the unique challenges and opportunities that emerged for her pedagogically in teaching mathematical inquiry. Finally, the chapter uses Harel and Koichu's (2010) principles of operationalising learning to summarise three overarching elements and their implications for adapting to a pedagogy of mathematical inquiry.

Introduction

Problems that students solve in mathematics rarely reflect the skills and understandings needed to address complex problems encountered in life. The well-structured problems in school mathematics contrast sharply with ill-structured problems, those for which the problem statement or method of solution is ambiguous. Nearly all problems in real-life are ill-structured and require negotiation to both define the problem and seek ways to address it. When mathematics strongly underpins the solution strategy of an ill-structured problem, then mathematical inquiry is used to address it.

"What is the best brand of bubble gum?" is a problem of mathematical inquiry. Students must debate what is valued in bubble gum to negotiate what is meant by "best", determine appropriate evidence to compare these valued qualities in bubble gum and justify decisions made during the solution process. For example, students may choose to assess several brands of gum on their elasticity, flavour, and ability to blow good bubbles by measuring each gum's stretching capacity after chewing, rate its flavour on a scale from 1 to 10, measure the diameter of a bubble and time how long it takes before a bubble can be blown. The mathematics in a problem such as this is rich, incorporating complex concepts from number, measurement and statistics, yet this type of problem rarely appears in mathematics classrooms. If students are to successfully transfer their learning from school mathematics to more authentic problems, they must learn to integrate and extend their mathematical understandings to incorporate twenty-first century skills such as collaboration, negotiation, persistence, critical thinking and creativity. Likewise, teachers must learn appropriate pedagogical practices to support students in inquiry.

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Several characteristics of school mathematics make inquiry more challenging than in other content areas, such as an assumption that mathematics is a discipline of pure truth. This chapter outlines key pedagogical characteristics of mathematical inquiry that differentiate it from problem solving and investigations. The benefits and challenges of teaching inquiry are discussed, many of which are unique to the mathematics classroom. Vignettes from teachers' classrooms are used to provide insight into the way mathematical inquiry is put into pedagogical practice.

Background

What is Mathematical Inquiry?

Mathematical inquiry is a process of solving ill-structured problems that significantly rely on mathematics in the solution process. Most problems in school mathematics are well-structured, where the problem is clearly defined and students enter the solution process with a limited number of pathways that would reach a successful solution. In contrast, an ill-structured problem is one in which the problem statement and/or solution path contains a number ambiguities which require negotiation to address the problem (Reitman, 1965). The solution of an ill-structure problem is typically not "right" or "wrong" but requires the solver to justify their conclusion, including the process used to reach it.

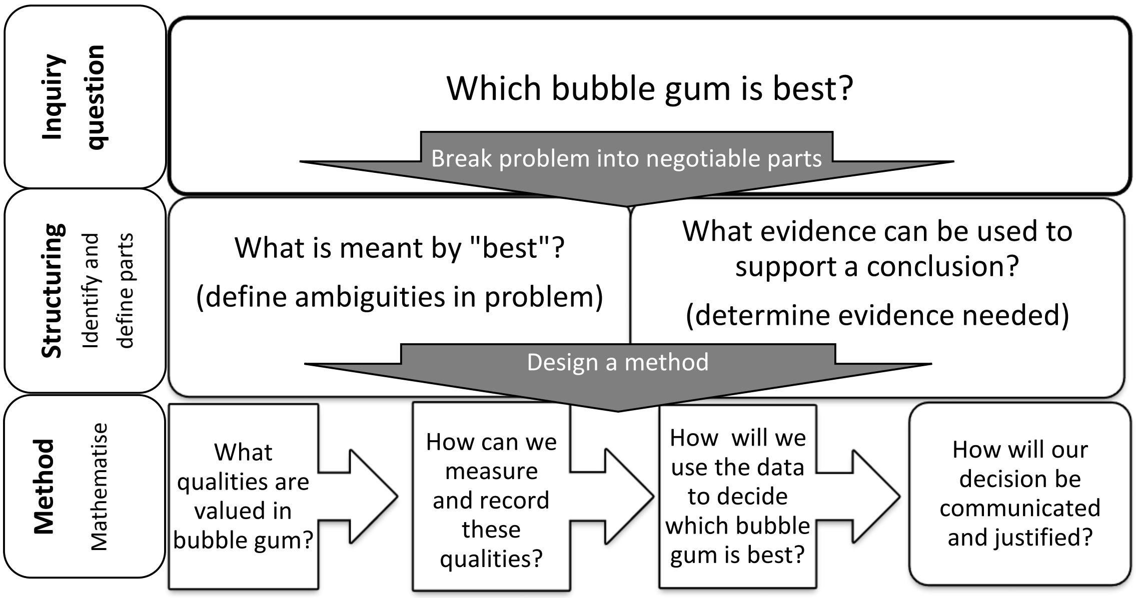

Figure 1: Negotiation of an ill-structured problem into a well-structured problem.

For example, in the ill-structured problem, "Which bubble gum is best?", a number of steps would be needed in order for students to come up with a solution to this inquiry question (Figure 1). This problem would require students to first negotiate what "best" means in the context of bubble gum. To do this, they would need to discuss which qualities they value in bubble gum (good taste, big bubbles, long lasting) and then find measureable ways to assess these qualities (e.g., rating scale, measure diameter of bubbles, time that strong flavour lasts). Students would need to design a method to determine and justify a solution with evidence. In this case, they might choose to run a series of experiments on different brands of bubble gum. The data they collect would need to be recorded and analysed in a way that would allow them to compare brands, then draw and justify a conclusion to answer the inquiry question. Finally, they would need to be able to communicate their solution and the process used persuasively to an audience. This process is non-trivial and it requires that the teacher be skilled in ways to appropriately scaffold students.

School mathematics and mathematical inquiry

For decades, there has been an increased emphasis on giving students more experiences with problem solving in mathematics (e.g., Schoenfeld, 1992). The aim was to augment students' work on the skills and concepts in mathematics and provide opportunities for them to apply their understandings to problems. Research has been clear that the mathematics learned in a well-structured environment does not necessarily transfer to a real-life problem (Mestre, 2005). There are four key ways in which traditional problems in school mathematics differ from authentic problems encountered outside of school (Allmond & Makar, 2010; Boaler, 1997; Schoenfeld, 1991).

1. *Consideration of the context*. School mathematics problems tend to neglect or over-simply the context or constraints in which the problem is situated. Unlike real life, if children are trying to decide the number of pieces of pizza that will be shared by several children in a school mathematics problem, it is not necessary for them to know the toppings that these children prefer or whether everyone is hungry and how many pieces they would want to eat. By ignoring the context, school mathematics tacitly encourages children to let go of their sense-making for the problem under consideration.
2. *Mathematising the problem*. Problems in school mathematics are stated in a way that has already translated any uncertainties in the problem into a question which is clear and unambiguous. In contrast, most problems in life are somewhat vague in nature and require the solver to make a number of decisions before and during the solution process. For example, in an everyday question like "Is it better to buy Chinese food or make your own?", the meaning of "better" could refer to cost, convenience or healthiness. Each of these would require a different method of investigation or may simply rely on opinion.
3. *Choosing mathematical strategies*. When children encounter mathematics problems in school, the problems tend to "cue" them to a mathematical strategy required to be applied. If the section of their textbook is entitled "Factoring quadratics" then students believe they are expected to apply this procedure to all of the problems they encounter in that section, even if this method is illogical for the problem solution.
4. *Attending to audience and purpose*. Finally, the purpose and audience from which a problem is derived is almost never mentioned in school mathematics. If children are trying to create the longest possible paper chain from an A4 piece of paper, they would need to know its purpose. If the paper chain is to be used to decorate the classroom, they would want to ensure that the loops are attractive and strong (i.e., links are cut uniformly and sufficiently wide). Knowing who their audience is for their justification allows them to target the evidence with an appropriate genre and level of rigour.

By de-emphasising context, sense-making, ambiguity, purpose and audience in school mathematics problems, children come to learn that mathematics is a set of disconnected skills, lacks meaning and relevance and is accessible only to a gifted few. Many argue that this has led to a sharp decline in students choosing to study mathematics beyond the compulsory level (Australian Academy of Science, 2006; McPhan, Morony, Pegg, Cooksey, & Lynch, 2008). A shift towards including mathematical inquiry into the curriculum and pedagogy of mathematics requires that teachers adopt and/or adapt a number of beliefs and skills which may be new to them.

The Pedagogy of Mathematical Inquiry

The target of pedagogical practice is student learning. Harel and Koichu (2010), working in mathematics education, define learning as follows.

> Learning is a continuum of disequilibrium–equilibrium phases manifested by (a) intellectual and psychological needs that instigate or result from these phases and (b) ways of understanding or ways of thinking that are utilized and newly constructed during these phases. (p. 118)

This definition aligns closely with Dewey's (1910; 1938) writings which emphasise how the process of inquiry, underpinned by these phases of disequilibrium-equilibrium, is needed for students to develop the reasoning and judgement to push beyond their unsubstantiated beliefs. To operationalise Harel and Koichu's definition, the pedagogy of mathematical inquiry relies on their three key principles for operationalising learning: Building understanding through construction of knowledge (Duality Principle), generating the need for concepts (Necessity Principle) and providing students with multiple experiences with reasoning to allow them to develop their understanding (Repeated Reasoning Principle). The process of inquiry is one which follows a fairly consistent pattern of phases with quite unpredictable pathways. In this chapter, we look at this process and focus on the pedagogical issues that arise when teachers engage students in inquiry. Using a case study to highlight these pedagogical issues within inquiry processes, we will return to Harel and Koichu's principles in the discussion to synthesise a pedagogy of mathematical inquiry.

Table 1

Phases and Pedagogies in Mathematical Inquiry.

Phase	This phase requires that teachers …	Pedagogical focus
Discover	Engage students in an issue of relevance to them and assist them in connecting the problem to one that can be address with mathematics	- Recognising the purpose driving an issue - Connecting context to mathematics - Negotiating ambiguities
Devise	Scaffold students through the process of translating a problem into one that can be addressed with mathematics (mathematisation) and planning a mathematical investigation, including recognising the evidence that will be needed to make and justify a conclusion	- Envisioning the process of inquiry - Mathematisation of problem - Focus on evidence needed - Develop classroom norms which encourage negotiation, meaning-making and habits of inquiry
Develop	Support students through their investigation, including enabling them to reason through unproductive pathways and revise their plans as needed. The need to represent ideas is at the forefront in the Develop phase and often requires explicit teaching as the need arises.	- Direct teaching of concepts needed - Developing mathematical modelling and representational fluency - Making argumentation and decision-making processes explicit
Defend	Provide students with an opportunity and reason to formulate a conclusion and communicate their findings convincingly to an appropriate audience.	- Explicitly linking purpose-question-evidence-conclusion - Valuing and communicating the process and limitations

Allmond, Wells and Makar (2010) developed a framework for teaching mathematical inquiry that takes into account the need to embrace the context and purpose of a problem, negotiate the underlying meaning and direction of an ill-structured question using mathematisation into one that can be addressed with mathematics, draw on a range of mathematical concepts and skills to plan and carry out a mathematical investigation and communicate and justify findings appropriate to the audience of interest. Their research has recognised the challenge of supporting students in developing a relevant question and plan to address it (Allmond & Makar, 2010; Fielding-Wells, 2010), the need for teaching children the language and structure of argumentation (Wells, 2010), and the challenges that teachers encounter in transitioning to teaching mathematical inquiry (Makar, 2010; Makar & Fielding-Wells, in press). The framework below was created to serve as a guideline for teachers to plan an inquiry lesson (Allmond et al, 2010), but is also useful to emphasise the pedagogical aspects of teaching inquiry in each of these phases.

Discover: Situating a question within a context and purpose

In the Discover phase, the children are introduced to the problem which will drive their inquiry. The word "problem" here does not refer to the task as in a textbook "problem", but rather to the larger contextual issue which drives the need to address a particular problem. For example, at the beginning of a school year, a teacher may argue for the need to re-design the arrangement of their classroom to maximise their needs. Using an inquiry question such as "What is the best layout for our classroom?", they would need to talk about what "best" could mean in this situation, including the reasons why a good layout assists with learning, specific requirements (e.g., space for sitting on the carpet, room to move between desks, easy and safe access to exits, constraints such as location of classroom computers and immovable furniture) and possible ideas for developing an approach. The Discover phase is often the "hook" that engages students in the overall unit of learning, but its purpose extends beyond engaging students. By connecting an issue to a problem that can be addressed with mathematics, children come to appreciate the relevance of mathematics and its value to them personally. They are also able to rely on their contextual understandings to build mathematical concepts and structures that underpin the problem and create a need for learning mathematics. The Discover Phase can also support students in dispelling a conception of mathematics as about right and wrong answers and begin to appreciate mathematics as a human construction with benefits and limitations in its application. Pedagogically, two areas of focus for the teacher in this phase could be a recognition of the connectedness of mathematics through a focus on the purpose and context which underpins the problem and assisting students in negotiating ambiguities.

Connectedness: Purpose and context. Because they were taught mathematics without an underlying purpose, teachers can sometimes focus on learning mathematics because it is written in the prescribed curriculum. They may forget to question the reason why the mathematical concept itself is useful. For example, if a topic such as "linear equations" is listed in the curriculum, teachers may have never had to consider what sorts of contexts lend themselves to linear models and why lines are selected as useful representations. Therefore, one aspect of pedagogy for teachers is to find ways to connect mathematical concepts to the world around them. In research which looked at over a hundred lessons of teachers' pedagogies as they transitioned from traditional mathematics lessons to teaching mathematical inquiry, Makar (2011) found that the greatest change was that inquiry provoked a need to *connect* mathematical ideas. What stood out in her research was the clear absence of connectedness in nearly all regular mathematics lessons.

Managing ambiguities. Because most problems in school mathematics are clearly stated, most teachers have little or no experience in managing ambiguities in problems. One of the most difficult aspects this raises pedagogically is the need to support students in defining ambiguities through negotiation. Mathematics teachers are typically inexperienced in discussing students' opinions, anecdotes and context-rich experiences as teachers of English may do when discussing a concept in a novel. Teachers must be experienced in seeing links between contexts and the mathematics which can be applied in these contexts. For example, if students are negotiating what the "best" design for a paper airplane is, the teacher must anticipate the mathematics underpinning "longest flight distance" and how it differs from the mathematics needed to test airplanes using "most accurately hitting a target" as the definition of "best". By seeing these mathematical connections, they can probe and scaffold students' ideas towards one that would be a productive avenue for mathematical learning. This may require that the teacher initially encourage students to just share their ideas and experiences on a topic without judgement and then slowly focus their conversation towards a mathematically viable question to investigate.

Devise: Planning an investigation

Once students have in mind an initial focus for their investigation, the Devise Phase is meant to provide them with time and support to plan (and possibly pilot) the investigation. Most investigations in school mathematics have already been "planned" by the teacher or textbook. This process, however, diminishes students' opportunities to translate their ideas and everyday understandings into mathematical processes. A "pre-planned" investigation does not allow them to wrestle with some of the challenges involved in this process of mathematisation. Because most problems in school mathematics are already set up to the point where students apply a mathematical procedure, they can develop a belief that authentic problems in mathematics have a clear sequence of steps. Inquiry, however, can be a fairly messy process and plans frequently do not progress as expected. In inquiry, students need to be able to persist through challenges and confidently reconsider unproductive directions without seeing this as a "failure". The ability to envision an inquiry process comes through repeated experiences with inquiry. Another aspect that may be new to students is the need to consider the evidence that would be needed for their analysis and for justification of their findings. These require a classroom culture which values negotiation and meaning-making and the sharing of partially-formed ideas through a constructivist view of knowledge acquisition. This phase poses a number of pedagogical strategies for teachers to engage with that may be new to them, including supporting students to envision how the inquiry process substantially differs from solving a traditional mathematics problem and developing a classroom culture which places importance on the process of understanding and not just a correct answer.

Focus on evidence. An important aspect of inquiry is a focus on evidence. In order for a teacher to support students in valuing evidence, there needs to be explicit teaching with students on the purposes of evidence and repeated discussions with students to keep the evidence they need in mind as they work. In research conducted with primary students, Fielding-Wells (2010; Wells, 2010) found that by explicitly focusing students on evidence during an inquiry, the quality of their planning in a mathematical inquiry improved significantly. A focus on evidence goes beyond asking students to "show their work", but includes an understanding of the value of evidence for both reasoning about the problem they are working on and to explain and justify their findings more persuasively.

Inquiry processes. The process of inquiry from a pedagogical perspective is one that is often new to teachers (Anderson, 2002; Crawford, 2000). Because most investigations in mathematics and science follow a clear solution path, the process of inquiry can often take both students and teachers by surprise. One element that is common in inquiry, for example, is that students may select a method to solve a problem that the teacher recognises (or may not initially recognise) to be unproductive. This can be difficult for teachers, however, who are accustomed to make learning easier for students. Researchers have found that teachers often blame themselves when an inquiry does not go to plan or if students run into difficulties (Anderson, 2002; Makar, 2010). Rather than redirect students away from this, it is often beneficial to allow students to find out for themselves how and why their solution strategy did not produce the desired outcome, and seek a more effective or efficient approach. This results in greater learning opportunities for students and develops their resilience if they are supported by a classroom culture that values the inquiry process.

Inquiry norms. A classroom culture of inquiry is one that requires explicit teaching. Students need guidance in understanding how to value meaning-making and working collaboratively, confidently share emerging and incomplete reasoning, negotiate and constructively critique ideas and see roadblocks not as "errors" but as opportunities for learning. Research on communities of practice (e.g., Goos, 2004) has shown the importance of repeated opportunities to engage in inquiry with explicit teaching in this process to allow students to successfully adopt a culture of inquiry.

Develop: Engaging in mathematical reasoning

The Develop phase provides time for students to carry out and revise their investigation plan towards a reasonable solution. The negotiation and re-negotiation as students encounter challenges often exposes shortcomings in students' mathematical understandings. This provides an excellent time for teachers to explicitly teach the needed mathematical concept. By embedding this teaching within the inquiry process, students come to value the relevance of mathematics over time and are able to situate their understandings in meaningful contexts. The ongoing need to discuss their evolving understandings and progress in addressing the inquiry question also provides opportunities for teachers to support students in developing argumentation strategies (Driver, Newton, & Osborne, 2000). A focus in this phase for students is to begin to move towards a conclusion and solution to their inquiry question. This may trigger a need for them to consider ways of using representations to reason with and communicate their findings. Pedagogically, these elements open up several ways in which to support students.

Opportunities for direct teaching. Very often, teachers new to inquiry assume that their role is always one of allowing students to direct their investigation. There are two keys aspects of inquiry in which teachers may choose to step in with direct teaching. One of these is when students are wrestling with specific mathematical concepts or skills. This is an ideal opportunity for students to practice a particular skill or deepen their understanding of a mathematical concept that has been exposed through the inquiry. It is not uncommon, for example, if students demonstrate understanding on a worksheet that they can calculate area or perimeter, but then in an authentic problem show that they wrestle with their understanding of these concepts. Direct teaching may be one way to assist them in connecting the formal mathematics with their application in context.

The other type of direct guidance that teachers may encounter is when an inquiry is losing momentum and students need assistance getting back on track. Inquiry is very different than discovery learning, where students are expected to "discover" the mathematics they need. Unlike discovery learning, which requires little input from the teacher during investigation, inquiry requires high quality scaffolding and expertise from the teacher in knowing how to balance when to step in and when to allow students to wrestle with challenging ideas.

Representations. In a traditional mathematics class, students create specific representations because they are instructed to do so by a textbook or task. The focus in these cases is generally the logistics of creating the representation rather than on understanding when or if it is useful for a particular context and purpose. This disconnect between the purpose of representations and their construction creates a gap in understanding and hinders students in developing representational fluency. While there are several representational forms which are used by convention in mathematics (e.g., linear graphs, pie charts, tables), the utility of a representation is in its ability to support and communicate reasoning for a particular purpose (Ainley & Pratt, 2010). In many cases, representations are created by the user and may not fit any convention, but provide a way to organise data or draft a conceptual framework. If inquiry is approached with a mindset that representations provide a model for understanding rather than as a conventional procedure to master, then students will develop much deeper understandings. For example, one aspect of graphs that students learn is to include a title of the graph and the axes. Another approach is for students to look for ways to improve how data can be understood by others. From this perspective, students learn that titles and labels on graphs are important for communicating meaning rather than simply one of many requirements for creating a graph.

Argumentation. Throughout an inquiry, students have multiple opportunities to discuss, negotiate and put ideas forward. This process becomes increasingly important as students are wrestling with concepts in the Develop phase. Because the focus is on consolidating their understanding in this phase, it is useful to direct students attention to the processes and reasoning behind reaching and justifying a conclusion. One aspect that has been noted in the literature on scientific argumentation (Kuhn Berland & Reiser, 2009) is that the level of understanding students engage with deepens as they move from working alone to understand (sense-making), to telling others their understanding (explanation) to finally working to convince others of the strength of their evidence and decisions which led to the results (persuasion). Providing students with opportunities to present their findings to an audience from which they anticipate questions is one way to help students seek better understandings as they anticipate presenting their solution.

Defend: Communicating and justifying a conclusion

The Defend phase is where students bring all of their work together to create and present their conclusion. In most school mathematics problems students seek a solution, but because the task has been carefully constructed for them, the process of linking their solution back to the question is less problematic than during inquiry. We have found that the process of connecting the parts of the problem—purpose-question-evidence-conclusion—is difficult for students (Allmond & Makar, 2010; Fielding-Wells, 2010) as they prepare to communicate their findings. In addition, most mathematics problems from school do not contain the issues of limitations that must be acknowledged in inquiry. These are therefore two particular areas for the teacher to place their pedagogical focus in the Defend phase.

Linking Purpose-Question-Evidence-Conclusion. As students prepare their conclusions, there is a need for the teacher to support them to ensure that their conclusion actually addresses the original problem and purpose. While being immersed in the solution process, it is easy for students to lose track of *why* they were solving the problem. Research has shown that it is very common for a mis-match to occur between students' conclusions and the problems they were addressing (Hancock, Kaput, & Goldsmith, 1992). For example, students may be finding out "How much do we grow in primary school (Years 1 to 6)?" and yet their initial conclusion may present the heights of children in the Years 1 and 6 rather than remembering to find the difference in the typical heights of these groups. As part of the conversation both with the whole class in general, and specifically as the teacher circulates to help individual groups, it is critical for the teacher to frequently ask students to articulate how their conclusion answers the original question being asked, and/or whether their solution addresses the larger problem purpose.

Students may find that their conclusion reveals that the problem was more complex than their question acknowledged. They may therefore conclude that they have "failed" their investigation because they cannot answer the question originally posed. They should not be discouraged by this, as it is quite common for people to over-simplify an issue with a fairly superficial question. The teacher can validate their frustration and remind them that by gaining a deeper insight into the original problem, they have made much more progress than answering what may now appear to be an inadequate question (showing them examples of simplistic questions in the newspaper may illustrate this). Teachers can encourage students to articulate how the original question was inadequate for the problem, given their new understandings of the issues and propose a better question that they could have asked.

Connecting their evidence they've collected to both their original question and their solution is another challenge for students. In finding out "Which grade level has the healthiest lunch?", one group of students collected data on the contents of their peers lunches in various grade levels. As they recorded this information, they may have also recorded the gender of the child. However, while the issue of gender would be an interesting issue, this data is irrelevant as the question being asked is not about gender. They may want to present the data on gender as evidence in their conclusion, prompting an opportunity for the teacher to re-visit the purpose of evidence. She may instead encourage them to present it as an "epilogue" to their inquiry rather than as part of the conclusion. Connecting all four of these elements—purpose-question-evidence-conclusion—takes multiple experiences with inquiry with the teacher frequently making explicit the importance of this connection each time students engage in mathematical inquiry.

Acknowledging limitations and possible new directions. Most mathematics problems in school end with the solution. When mathematics is applied to an authentic context, however, there are always limitations. In order to reduce a complex, open-ended problem that can be addressed with mathematics (especially mathematics that is accessible to the level of the students), decisions will have been made to simplify the problem. Just as a map provides only one perspective of a terrain, the solution path that students take will provide a simplified version of a original problem. For example, in the problem above, "How much do we grow in primary school?", it would be ideal for students to have data on their own heights in Year 1 to compare to their Year 6 height. This data is unlikely to be available for all students, however. By finding the difference between a typical Year 6 height and a typical Year 1 height, students can provide an estimate of how much students may typically grow in this period. However, there are limitations that should be expressed. The students in Year 1 may not be a good representation of the Year 6 students' heights from 5 years ago. If students try and generalise their findings to all Year 6 students, they will need to acknowledge that their heights may not be representative of the larger population of students at this age. Mathematising problems will almost always create a simplified version of the original problem. By acknowledging and being explicit about this, it provides an opportunity to break down perceptions that mathematics will provide a clear and unambiguous solution to complex problems encountered in life or work.

The discussion above raises only some of the issues that teachers encounter pedagogically in teaching mathematics through inquiry. To illustrate a few of these issues, a case study is provided below of a teacher early in her journey of learning to teach mathematics through inquiry. In this vignette, many of the pedagogical issues above emerge and are discussed.

Case Study

Background and Method

The case study presented below comes from a classroom of a multi-age classroom of students in Years 6 and 7 (aged 10-13). The teacher had been teaching mathematics through inquiry for close to a year in these episodes as part of a larger Design-based research (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) project aimed at understanding teachers' changing experiences as they adopted inquiry-based pedagogical practices in mathematics (Makar, 2010). The school was situated in a low socio-economic area in rural Australia. Mathematical inquiry was a whole-school initiative, so students had been participating in the project along with the teacher for about a year. Because the school experienced high levels of attrition due to a fairly migrant student body, there were also students in the class with little experience learning mathematics in this open-ended environment.

Two units are presented below. In each unit, at least two lessons were videotaped, with a field notes taken by the author. The teacher was interviewed following each unit to assist in understanding their evolving experiences. These interviews were audio recorded and transcribed. A content analysis of both the video and interview transcript (Flick, 2009) allowed the author to locate salient episodes to both present the sequence of lessons and to illustrate particular issues that arose. In this pair of units, a key issue was raised about how students demonstrated their understandings both during the first unit and as it was transferred to the second unit.

Which Bubble Gum is Best?

April Frizzle[1] modified the unit *"Which Bubble Gum is Best"?* from a published investigation (Friel & Joyner, 1997). She chose the unit because she thought it would engage students, help them to connect mathematics to everyday contexts and give them experiences with a diverse set of measurement concepts and tools. An overview of the key lessons in the unit are given below (Table 2).

[1] A pseudonym chosen by the teacher in this case study, Ms Frizzle is the inspirational teacher in the popular children's book and television series *Magic School Bus*.

Table 2

Planned lesson sequence in Which Brand of Bubble Gum is Best?

Lesson	Phase	Lesson Focus	Mathematical focus
1	Discover	• Understand the problem and context; • Make informal links between the problem and the mathematics	• Develop the need to measure and quantify characteristics; • Strengthen mathematical language
2	Devise	• Develop measures that align with desired characteristics; • Pilot measurement methods; • Design data recording form	• Connect quantitative measures with qualitative characteristics; • Explore measurement tools; • Become aware of data structures (for data recording)
3	Develop	• Collect, record and organise data • Conduct analysis of findings and record preliminary conclusion	• Data collection and representation
4	Devise	• Complete data representations and present findings to the class; • Reflection	• Data representations • Bias in data collection • Ways to improve

Discover Phase

In the unit, students spent the first lesson orienting themselves to the context and informally making links between the context of bubble gum and the mathematics involved. In the initial lesson, students expressed their opinions about what makes for good bubble gum and then brainstormed ways to classify and describe valued characteristics of bubble gum. They selected 3-4 characteristics that they thought they could quantify. This initial time for students to immerse themselves in the context was critical for them to (1) engage with and come to a common understanding of the problem under investigation, (2) move beyond the initial excitement of the context so they could focus on the mathematics in the task and (3) familiarise themselves with the elements of the context that could be mathematised – the key process of converting an ill-structured problem into a well-structured one that can be investigated with mathematics.

This initial lesson is often overlooked as teachers are eager to get to what they consider to be the "real mathematics" of the unit. However, this period is critical for students to gain a deep enough idea of the problem and context to make meaning out of the mathematics in later lessons. It also works as a "hook" to engaging students with the problem.

Devise Phase

In the second lesson, Ms Frizzle worked with each collaborative group to mathematise the problem. That is, to develop ways to quantify desired characteristics of bubble gum so that they could compare different brands. Students found a number of characteristics to evaluate and with support and skilled questioning from the teacher, generated ideas to quantify or measure each characteristic. A list of the ones students generated and their ideas for quantification are given below (Table 3). These were reviewed during data collection and revised once they found them to capture the desired characteristics.

Although students had worked with all of the measurement instruments before, she found that they had never had to connect the instrument with a purpose for selecting it (why would you use callipers instead of a ruler to measure the diameter of a bubble?). Rather than direct them to the instrument, Mrs Frizzle gave them an opportunity to experiment with a variety of tools available in the room (e.g., measuring tapes, pedometers, stop watches, etc.) so they could learn to connect an instrument with the context and purpose in which it would make sense. It also allowed them to recognise the shortcomings of different instruments. She found that during the initial discussions, students had naïve ideas about the nature of measurement that likely arose from always being given only "ideal circumstances" in which to measure objects in their mathematics textbook.

Table 3

Measureable Characteristics of Bubble Gum.

Characteristic	Measure	Description
Size of the bubble	Diameter of bubble	Use of callipers (or "eyeball" with a ruler), the diameter of a bubble blown by a student.
Flavour quality	Rating	Students devised a 5-point rating scale to describe the quality of the flavour.
Flavour duration	Time (minutes)	Time from first "chew" until the gum had lost most of its flavour.
Elasticity	Length	Difference between length of unchewed stick of gum (cm) and maximum stretched length of gum after being chewed.
Others?		

Develop phase

When it came time to collect the data, Ms Frizzle provided each group with a paper plate and three types of bubble gum to test. She reminded them to think carefully about the order of their tests. For example, if they needed to measure lengths or any initial aspects of the gum before it was chewed, they should plan accordingly (once gum is chewed it cannot be returned to its original state to be measured). She found that they tended to ignore this advice, however, and just jump in and start chewing, then think later about the measurements and data.

By the end of the lessons, students had collected their data, although many of them struggled with either being careful about its collection or what to do when they ran into problems. Ms Frizzle also found it difficult to see students make poor choices about their data collection and was worried that they hadn't gained the mathematical experiences she had intended.

Defend phase

After the previous lesson, Ms Frizzle was disappointed. She wanted to complete the unit, however, and gave each group a brief opportunity to show the class the data they had collected and represented (collating the data for the whole class on the board) as well as their conclusions to the inquiry question. In most cases, she found that students ignored their data and relied on anecdotal experiences to make their conclusions. She had intended to give students an opportunity to make suggestions about what they had learned and what they might improve next time. Her experience with the unit, however, momentarily dampened her enthusiasm and she decided not to spend time reflecting on the unit at that point in time.

Follow up Interview

In an interview following the unit, April Frizzle expressed her reaction to the unit and areas that she felt could be improved for next time. The researcher started by asking her to reflect on her experience with the Bubble Gum unit.

April: Well, the disaster of the bubble gum experiment!

Int: Tell me what you mean by that, why you thought it was a disaster.

April: Well, it wasn't. Initially, it was really good, because the pre-discussion was really good, and what we thought we were going to do – the direction we were going to go in I thought was fantastic, and there was a lot of mathematical talk, but when we actually introduced the bubble gum to the equation, it kind of went silly, and I should have said, "Okay, everyone can have one piece of bubble gum first (laughs). That would have been better! But once they started chewing bubble gum, they forgot about all the pre-stuff and it was very very difficult to bring them back on task. And I think that that was my big concern – bringing them back, dragging them back from the joy of the bubble gum. …

There was an acknowledgement from April that she hadn't given her students an opportunity to get the excitement of the bubble gum out of their system before settling into the mathematics. This opportunity to "explore" the bubble gum in the Discover or Devise phase could have addressed this concern and April was hard on herself for not realising that.

April: Yep. Free play. I should have known.

Int: [to] get it out of the way?

April: But I just thought, "Oh, no. They'll be alright. They can eat the bubble gum at the end."

April's concerns with how to balance her vision of the lesson and taking more control of the direction of the lesson with giving students an opportunity to develop and guide how they collected and interpreted their data is one that is common for teachers teaching inquiry (Makar, 2010). She didn't want to squash the students' enthusiasm or cut short their explorations. Because April has a passion for teaching art, she recognised the importance of the creative process and allowing students to learn from their experiences. Although she embraced and was excited about the open-endedness that an inquiry-based approach could bring to mathematics, she was unfamiliar with how these ideals could be apply to mathematics lessons. April recognised a lot of positives in the lesson as well as how having better access to resources would have assisted her.

Int: You said you thought the first part went well. Tell me [more] about that.

April: Yeah, well, what they thought they were going to do, like, how they were going to record their results and things, and what they were going to test. So they were testing for the shape of the bubble gum, they were testing for its elasticity, they were testing for how long it would take to lose the flavour. So there was a lot of measurement involved, and I thought that was really good. And then when they talked about how they would do that – that was really good, too. But then when we came to actually do it, we didn't have all the resources on hand that we found they needed, and so we probably should have planned a little bit better as to, okay, what do we need to do this, and make sure that that was in the room.

Int: Those things are available.

April: Yeah, because we lost time, we lost interest because they (untrans). I think we really needed to have all those resources at hand, yeah.

It was common for the teachers to blame themselves when things didn't go to plan (Krajcik et al, 1998; Makar, 2010; Marx et al, 1994) and this was no different for April when she said she "probably should have planned a little bit better … and make sure that that was in the room". It was also fairly common for the teachers to focus on student engagement during the lesson as a tremendously motivating aspect for both her and her students (see also Kennedy, 2005).

April: We went on tangents (laughs) all over the place, we were going. And they were – I mean, they were excited about what they were doing, and they were engaged. And I think mostly they were on task. It's just that once they started chewing, that sort of took over everything else. But I thought they were pretty good.

One other area that April wished she had spent some time on was students being able to reflect on the unit.

April: We really needed to reflect on what we did, and we really needed to say, "Okay, what did you get out of it?" But we kind of left it, and then went on to something else. … We needed to do that reflection there and then. And we kind of ran out of time. So had we done that, I think they would have realised that what they were testing for was fine, but they needed to have the framework drawn up already so they could record very well. They needed to have the equipment readily available so they could do everything they wanted to, and then they could make interpretations – and I don't think we got to that stage. So we need to finish off things better, and we didn't do it with that.

April finished off the interview with a recognition that students had made some progress on the basic concepts of the unit and it has also allowed them to transition well into a science investigation on moulds that they did later in the term. She was pleased that both she and the students were getting better at inquiry processes and she was happy to see them progress.

April: [I liked that] it linked in to the science unit that we're doing with the mould, and making predictions and everything about that, and recording the information. … They're getting better in working in groups, and they're getting better at organising themselves. I don't care if they get it right or not, as long as the process, they're learning from that. And that's what they seem to be doing, so, yeah. But you sort of have in your mind as a teacher that there should be a right thing, but with this, there is no right, as long as they can tell you something about it – that's the big thing. So, yeah, I guess, although I felt like it was "laaa!" (laughs) and it was chaos, they did learn quite a bit from that, yeah.

The interview was conducted about two months after the unit, so the distance from it had given her an opportunity to reflect and see a number of positives in the unit. Possibly, the chance to talk about these aspects motivated her to design the next unit, which seemed to be an opportunity for her to revisit the experience of the Bubble Gum unit in a similar context.

What is the Best Orange?

The next term, Ms Frizzle decided to try a similar inquiry task that would enable the students to go through the process of devising and testing measurable qualities of an orange. The unit was built on the Bubble Gum unit as well as drawing on ideas from a published unit on lemons (Smith & Peterson, 2006). She decided that by using oranges, students would be less distracted than they had been with the Bubble Gum unit yet still be able to employ similar opportunities for them to engage in higher order thinking. The structure of the hypothesised lesson sequence was similar to that in the Bubble Gum unit of attending to one phase of inquiry for each lesson (Table 2). She entered the unit with the idea of having a "fresh start", determined to have more success this time.

Discover Phase

The aim of the Discover phase was to first orient students to the problem and inquiry question and them to give them an opportunity to brainstorm ideas for testing an orange. She began by asking them to estimate the weight of an orange and suggest what the weight of an orange might suggest about an orange's characteristics (e.g., its juiciness). The inquiry question for the unit was, "What is the best orange"? The rest of the lesson asked students to brainstorm characteristics that could be compared between different varieties of orange and also a measure that could be used as evidence for those characteristics. Ms Frizzle noticed that this time, students appeared more confident and focused in what they needed to do.

Devise Phase

Because the students had made more progress in the previous lesson, Ms Frizzle had them go straight into devising their plans for measuring characteristics. She was pleased to see that they came up with quite a number of ideas. She focused them on ensuring they had three aspects written down for each characteristic they were measuring: (a) the characteristic of the orange, (b) the measurement for that characteristic and (c) how the evidence would be recorded. Once groups had begun the process, the class came together to share their ideas for quantification of characteristics of oranges they had chosen to assess (Table 4, note that not all areas are complete). By sharing these ideas with the whole class early in the process, it enabled groups to generate new ideas if they were struggling or to get feedback to circumvent unanticipated problems. From this list, students discussed how data could be recorded for individual oranges. Ms Frizzle was surprised that so many students saw a graph as a recording devise for data rather than a way of representing data for communicating interpretations. Some suggestions also indicated that students were recording measurements because "they could" (e.g., height) rather than linking it to a valued characteristic of an orange.

Table 4

Some Measureable Characteristics of Oranges

What are we measuring	How do we measure that?	How do we record the evidence?
taste		Table/graph/tally marks
spins	Timer/stopwatch	graph
height	Measuring tape	Bar graph
juice	Measuring cup	graph
smell	Smell detector	Rating scale
Weight	scales	

Develop phase

Ms Frizzle handed each group one orange per person to test, reminding them to pay attention to the order of the measurements. For example, once they cut an orange open and extracted its juice, it would be difficult to go back and find the weight or diameter of the orange. She was pleased to see students more attentive this time, appearing to have learned from their experiences with testing the bubble gum in the previous unit. Unlike the previous unit, the students appeared to be more confident in the process and carried it out more independently. She found that in her discussions with groups as she circulated, the questions students asked were more focused on the mathematical concepts and representations rather than organisational logistics. She was able to use richer forms of questioning to get at concepts to deepen students' understanding ("What does this mean? How can you demonstrate that idea? What evidence do you need to justify it? How will you know what this graph represents in two weeks time?"). Conversations often revolved around the validity of results and whether graphs were the only way (or best way) to represent the data. She was pleased that the majority of her time was spent this time with individual groups supporting them to make effective and meaningful choices in representing their results.

Defend phase

The students presented their findings to the class and their peers were invited to ask questions. Ms Frizzle noticed that they had little experience with asking relevant and productive questions of one another and made a note that she would make this more of a priority with her class next year early on. The presentations further raised issues of the problematic nature of recording data like smell or taste that were more subjective in nature. She was pleased that the students engaged in these conversations and could see the experience had a positive impact on dispelling the notion that mathematics was only about right and wrong answers.

Follow up interview

April's interview following the unit demonstrated that she had been pleased with the overall success of the unit this time.

> April: I just think comparing this unit of work to the last unit of work, I didn't have to guide them as much as I would normally. And I think that was a lot better. They sort of took a lot from the … bubble gum lesson, and then adapted that to the oranges thing, although they found the oranges boring in comparison to the bubble gum! But I think it was more of a flavour issue than anything. So, yeah.
>
> Int: Not such a bad thing.
>
> April: No, that wasn't such a bad thing. They're getting better at recording their information without me having to say "What do you think about this?" So they were pretty good at that. But we need to still be able to tie it all together at the end a little better. It's still sort of higgldy-piggldy at the end. But I think they, they have got a lot out of the lessons. … They've probably learned more in that session than they have in anything else, because their testing shows that they have, so. Yeah, no, I think it's been good.
>
> Int: … I noticed that in the bubble gum one, they did eventually get pretty good at coming up with what you would measure, but that took a lot of work on your part to scaffold.
>
> April: Yep.
>
> Int: What did you find this time, because it's actually (knowing what to measure) a really hard concept to get across.
>
> April: I think this time it was a lot easier. Because they had that in their mind about before, so they sort of knew what I wanted from them, without me having to specify that I want this, that and the other. And I found that I wasn't asking as many questions this time, and if I was asking questions, it was more on how to fine tune it. Because they had the idea of what they wanted, but just getting that down pat was probably a little bit harder. So that was good. And I think that they were much more on task in this one. Like they seemed to be keen to get it all done and were working well together. And their teams were working well together. So that was good, too.

April's comments suggested that she was starting to see that the "disaster" that she had described as the Bubble Gum unit had actually allowed students to gain a lot of experience that they applied to the Oranges unit. Her expectations in the Bubble Gum unit were that students would more visibly attend to and demonstrate their understanding of measurement with an organised and focused experiment. What was evident in the Oranges unit, was that the Bubble Gum unit had played an important role in enabling students to envision the process, experiment with measurement concepts and develop an initial awareness of the problematic nature of turning qualitative characteristics into quantitative measures. These experiences were quite different than experiments in which they had been told what to measure and how, but never had to consider what the purpose of the measurements were or that measurements often had shortcomings in capturing qualities intended. In the interview, April articulated that she could see some transfer between the two units.

> Int: In this one, it's almost like the Bubble Gum investigation … when you went through the Orange one, they could draw on that. …
>
> April: Yep. And they were doing that. They were telling me that. "Oh, I remember doing this with the bubble gum." And I'd say, "Well, will that work with this one?" "Oh, no, because we were testing stretchiness." … They brought in similar elements. Okay, if you want to test the flavour, how are you going to record that information? And they were good at, you know, doing the line thing (rating scale) and gauging that, so, no, they were pretty good. We still struggle with our graphing and why we're graphing, because they tend to think a graph will be great for everything, and that's not always the case, so we'll keep battling on. But I do really think that they've come a long way in a year.
>
> Int: It's taken a year, though, hasn't it?
>
> April: Oh, definitely. Yeah. …
>
> Int: The fact that the bubble gum and orange were so similar, it really stood out, I guess. That they don't just need one experience, they actually need the second one to be able to draw on the first one.

Finally, April commented on the way that the students' experiences in inquiry over the year had materialised in their beliefs about the nature of mathematics and how that changed the way they were willing to approach problems now.

> April: I think [students] are a lot less worried about making a mistake than they ever were. Before it was, "Oh, I don't know that, because I don't know the right answer." But now they understand that there *is* no right answer. It's just a matter of getting to the end, and the journey along the way. And I think that's taken a year.

April's reflection on the Oranges unit suggests that her experience was a sharp contrast to that in the Bubble Gum unit. However, it is also clear that the Bubble Gum unit, while not apparent at the time, had a strong impact on students' developing understanding of measurement and comparisons. This raises a critical pedagogical issue about the pedagogies teachers use to develop students' understandings over time.

Discussion and Implications

The pedagogy of mathematical inquiry aligns well with the ideals promoted by reform-oriented research in teaching and learning mathematics: Problem-solving, communication, reasoning and connections (National Council of Teachers of Mathematics, 1989). It engages students intellectually regardless of their performance in traditional mathematics tasks, draws on rich mathematical concepts, is situated in authentic problem contexts, provides them with a range of experiences which require them to make meaning of their understanding, encourages students to negotiate and collaborate with their peers and promotes a student-centred classroom. Although these ideals have targeted the teaching of mathematics for some time, these elements are rarely found in mathematics classrooms (Boaler, 1997; Stigler & Hiebert, 1999; Hollingsworth, Lokan, & McCrae, 2003; Clarke, Keitel, & Shimizu, 2006). What is unique about mathematical inquiry is that these pedagogies are not only recommended, but necessary. Without opportunities for diverse learners to engage in rich mathematics, there would be no place for mathematical inquiry in the schools of today which demand differentiated learning and rigorous content. The nature of addressing an ill-structured problem necessitates negotiation and collaboration with context-laden reasoning and meaning-making.

Inert knowledge is one of the key challenges in mathematics education (Bakker & Derry, 2011). There is often an assumption that what students learn in a structured environment will transfer to less-structured contexts (Metre, 2005; Nunes, Schliemann, & Carraher, 1993; Saxe, 1988; Schoenfeld, 1991). Transfer literature has been clear that this assumption is highly flawed. Dewey (1910) explains that problems do not come pre-labelled with the concepts needed to address them.

> There is no label on any given idea or principle which says automatically, “Use me in this situation”—as the magic cakes of Alice in Wonderland were inscribed “Eat me.” . . . If one is not able to estimate wisely what is relevant to the interpretation of a given perplexing or doubtful issue, it avails little that arduous learning has built up a large stock of concepts. (pp. 106–107)

With student learning at the forefront of pedagogical practice, a framework for operationalising learning forms of the basis of the discussion and implications below which address the issue of inert learning that Dewey expresses. Harel and Koichu’s (2010) put forth three guiding principles as foundational to operationalise learning (p. 118):

- *The Duality Principle*: Students develop ways of thinking through the construction of ways of understanding, and the ways of understanding they produce are determined by the ways of thinking they possess.
- *The Necessity Principle*: For students to learn the mathematics we intend to teach them, they must have a need for it, where ‘need’ here refers to intellectual need.
- *The Repeated Reasoning Principle*: Students must practice reasoning in order to internalise desirable ways of understanding and ways of thinking.

In this discussion, these principles and the case study above are used to discuss and highlight important implications for teaching and teacher education that are emerging in research on the pedagogy of mathematical inquiry.

Duality

Duality emphasises a constructivist perspective on the teaching and learning of mathematics (Confrey & Kazak, 2006). Rather than telling them the mathematics they are to learn, mathematical inquiry creates a situation in which the teacher can support students in constructing their understanding. As in the case study above, teachers often find it difficult to observe students as they attempt solution strategies that the teacher may consider as non-productive. Following the Bubble Gum unit, the teacher expressed disappointment that the students did not appear to have developed the understandings she was expecting. She therefore deemed the unit a failure and did not complete it. This is a common response when teachers observe their students constructing understanding and interpret as ineffective the indirect or apparent (to the teacher) non-productive pathways that students take (Makar, 2010).

More commonly, teachers work to avoid this "struggle" that they see their students undertake and try to simplify or redirect students away from non-productive pathways before students proceed. Boaler (1997) points to the conflict that teachers often feel when they see their students struggle or wrestle with ideas as they engage in constructing their understanding.

> Teachers gave the students these 'handy hints' or rules to make mathematics questions easier, more straightforward, for students. The teachers understood the mathematics they were talking about and from that base of understanding the rules appeared to be helpful to them. But the students did not understand the rules they were learning or the way that these rules related to the different situations they encountered. They did not locate the rules within a broad mathematical framework and they did not develop a real sense of what they meant. … They view the procedures as abstract rules to be learned and to which they should adhere. Rules may be easy to learn, but difficult to use if they have not been placed within a wider sphere of understanding. (p. 26-27)

Mathematical inquiry requires that teachers adopt a pedagogy in which they can balance allowing students to construct their understanding with the need to support and scaffold that learning. This is in contrast to "Discovery Learning" in which students are not provided with this support. Confrey (1991) emphasises that it is not that teachers want students to "uncover" what the teacher wants them to find, but to reconceptualise the problem in a way that builds on their own understandings about the world.

Necessity

If you ask teachers why they teach a particular topic in mathematics, a common response is that they teach it because it is in the curriculum. Because most teachers have been taught in a traditional pedagogical approach which does not encourage them to question the relevance of mathematics (Ball, 1996), they are often inexperienced at finding specific examples of the "need" for a mathematical concept. At the heart of the case study was the teacher's aim for students to generate a need to measure and to learn multiple ways to operationalise this need. Engaging in the process of mathematisation is an important avenue for students to generate the need for mathematical concepts. This is in sharp contrast to the norm of teaching mathematics, where the teacher presents a concept to students without making clear its relevance or need for addressing an authentic problem (Stigler & Hiebert, 1999).

Inquiry is embedded in a need to resolve doubt (Dewey, 1910; 1938; Peirce, 1923/1998). During inquiry, students internalise a problem and make it their own. This internalisation of a problem is articulated by Confrey (1991) as the "problematic". If teachers can use mathematical concepts as a way to address human needs through a problematic, then they will begin to support students to (1) see the relevance and need for mathematics for everyday life and work (McPhan et al, 2008), (2) situate their understanding in a meaningful context (Lesh & Zawojewski, 2007) and (3) shake the belief in mathematics as a discipline with right and wrong answers and one "correct" solution to every problem (Boaler, 1997). Researchers have long argued that teachers' previous experiences and beliefs about the nature of mathematics underpin their pedagogy in teaching mathematics (Thompson, 1992). "Ideas have consequences. One's conception of what mathematics *is* affects one's conception of how it should be presented. One's manner of presenting it is an indication of what one believes to be most essential in it" (Hersh, 1986, p. 13). If teachers' beliefs about mathematics is steeped in an epistemology in which mathematical knowledge is grounded in "learning terms and practicing procedures" (Stigler & Hiebert, 1999, p. 41), then seeing the need for mathematics around them may be difficult (Philipp, 2007). Like the quote from Boaler (1997), they shortcut this process and go straight to the abstract procedure.

In her research with teachers learning to adapt their teaching to an inquiry approach, Makar (2011) found that in traditional mathematics lessons, the teachers rarely connected the mathematics they were teaching with its relevance beyond the classroom, or even beyond the lesson they were teaching. Makar's research showed that even with teachers new to inquiry, connectedness was an element that was immediately apparent in their lessons. This suggests that pedagogically, an inquiry approach by nature connects the need for mathematics to a context and/or problem in which it is situated.

Repeated Reasoning

Finally, Harel and Koichu (2010) argue that to operationalise learning in mathematics, students must have repeated opportunities to reason in meaningful ways. Pedagogically, this implies that teachers must see learning is not a single "event" or lesson, but an ongoing iterative process that requires multiple learning experiences in which students can apply and build their understandings. The ability to investigate an idea and produce a convincing argument to communicate their understanding requires that students have time and multiple opportunities to engage in these practices (Koedinger, 1998). The case study above made this point quite clearly. If the teacher had engaged students only in the Bubble Gum unit, she would have concluded (as she did in the interview) that students had really not learned anything. Her choice to develop the Oranges unit was not to extend students' understandings or to try and assist them to apply their learning from the Bubble Gum unit to a new context. She had assumed that nothing was gained from the Bubble Gum unit and that she needed to "try again" to build their understanding of the need for measurement by removing the temptations of gum. It was only when she observed her students draw on their experiences with the Bubble Gum unit that she could recognise that they had in fact learned quite a lot from that unit. The students articulated quite explicitly that they were using concepts from the Bubble Gum unit in designing their measures of the characteristics of oranges. In this way, they demonstrated that they were developing an understanding of the need for measurement as well as how to meaningfully mathematise the qualitative characteristics valued in oranges (taste, juiciness) into quantitative measures. The teacher also noted that students were much more independent and confident in their work on the oranges unit. This enabled her to probe and deepen their understanding to build much richer concepts of the mathematics of measurement.

McGowen and Tall (2010) used the term "met-before" to describe the opportunities previous experiences create for learning and building confidence to manage problems in new contexts. They defined a met-before is "a mental structure that we have *now* as a result of experiences we have met-before", arguing that "met-befores enhance the chances of making sense of new ideas, increasing the possibility of achieving the goal of conceptual understanding" (p. 171). The transfer literature supports this conception of learning. Rather than see learning as an object that is acquired in one context and then moved intact to another, transfer advocates conceptualise learning "as a cognitive *state* the learner enters or forms at the moment, involving the activation of multiple resources ... [with] the learner entering or forming a similar state later in a different context" (p. 93). Fisch, Kirkorian, and Anderson (2005) argue that transfer of learning is promoted by enabling students to meet a concept across multiple contexts.

> Through this sort of experience, the mental representation of the underlying content is forced to adapt in subtle ways to each new context, yielding a representation that gradually becomes more detached from the specific contexts presented, so that t can be applied more easily in new situations as they are encountered. Indeed, Betterworth, Slocum, and Nelson (1993) have gone so far as to argue that presenting only one example provides no basis for generalization and transfer. (p. 382)

The synthesis of "met-befores" with this perspective of learning transfer puts forth a strong argument why students require multiple opportunities to engage in reasoning that promotes understanding of a concept. Pedagogically, this perspective is operationalised by teachers through explicit attention to the idea that students learn through iterative experiences rather than in a single lesson.

The pedagogy of inquiry assumes that students are working in an environment in which they are expected to question assumptions, negotiate meanings, share emerging ideas and collaborate with their peers. Teachers have an important role in building these socio-mathematical habits of working in an inquiry-based environment, "built up under the influence of a number of particular experiences" (Dewey, 1910, p. 145). A classroom culture of inquiry critically requires multiple opportunities for students to develop habits and norms with the guidance and explicit support of the teacher (Cobb, 1999; Goos, 2004).

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