

ASSESSMENT FOR COMMON MISUNDERSTANDINGS – INTRODUCING THE BIG IDEAS¹

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Purpose:

Scaffolding student learning is the primary task of teachers of mathematics, but this cannot be achieved without accurate information about what each student knows already and what might be within the student's grasp with some support from the teacher and/or peers. This requires assessment techniques that expose students thinking, but it also requires an interpretation of what different student responses might mean and some practical ideas to address the particular learning needs identified. This is particularly important in relation to a relatively small number of 'big' ideas and strategies in Number, without which students' progress in mathematics will be seriously impacted. The tools comprise a number of easy to administer, practical assessment tasks designed to address a key area of Number at six key levels of schooling – the end of Foundation, the end of Year 2, the end of Year 4, the end of Year 6, the end of Year 8 and the end of Year 10. A hierarchy of student responses is identified for each task and, for each of these, an interpretation of what the response might mean is provided together with some targeted teaching suggestions.

Background:

The tools are based on a series of highly focussed, research-based *Probe Tasks* which were developed for teaching purposes at RMIT University² and subsequently used in the context of the *Supporting Indigenous Student Achievement in Numeracy Project* in the Northern Territory to identify learning needs in Number. To support teachers in remote schools interpret student responses, identify learning needs, and choose developmentally appropriate tasks to address those needs, advice in the form of a *Probe Task Manual*³ was prepared. The advice was prepared on the basis of the broader research literature, 'mainstream' student responses derived from Victorian classrooms, and the responses of a small sample of Indigenous students who participated in the NT study. The *Probe Tasks* and the *Probe Task Manual* are currently being used by teachers in Northern Queensland to identify and address student learning needs.

The tools presented here draw on the *Probe Tasks* and the *Probe Task Manual* but include a number of additional tasks and resources which have been organised to address some 'common misunderstandings'.

A Note on 'Common Misunderstandings'

The following beliefs and/or practices would be fairly widely recognised by most teachers of mathematics:

- multiplication makes larger, division makes smaller;
- when subtracting by place-value parts, find the difference;
- when multiplying (or dividing) by powers of 10, zeros are 'added' or 'crossed off' to find the answer;
- the larger the numerator or denominator, the larger the fraction (conversely, the smaller the numerator or denominator the smaller the fraction);
- the longer the string of decimal fraction parts the larger (or smaller) the number;
- when adding/subtracting fractions, you add/subtract denominators on the grounds that 'what you do to the top, you do to the bottom';
- when multiplying or dividing decimal numbers by powers of ten the decimal point moves to the right or left;
- the tenths, hundredths, thousandths etc live to the left of 0 on the number line;
- when dividing you don't need to record 0 as it means nothing (eg, 138 recorded as the answer to 6540 divided by 5);
- to change a fraction into a decimal, multiply by 100 over 1; and
- two negatives make a positive.

¹ <http://www.education.vic.gov.au/studentlearning/teachingresources/maths/common/default.htm>

² Siemon, D. (2003). *Probe Task Folio*. Unpublished Teaching Resource. RMIT University (Bundoora Campus)

³ Used with permission from the Australian Department of Education, Science and Technology who funded the *Supporting Indigenous Students Achievement in Numeracy Project* (2003-2004) under the National Literacy and Numeracy Strategies and Projects Programme. The project was administered and supported by NT DEET, the Office of Catholic Education Darwin, and conducted by Professor Dianne Siemon from RMIT University.

While some of these might be regarded as 'misconceptions' and others as 'helpful rules' introduced by teachers and/or texts, they have two things in common. They both work some, if not all, of the time depending on the particular context in which they are exercised, and they both serve to reinforce the view that school mathematics is about learning and applying fairly meaningless rules or procedures.

A Thesaurus search on 'misunderstandings' produces: "mistakes, quarrels, mix ups, errors, misconstructions, confusion, misinterpretations, misapprehensions...".

A search on 'misunderstand' produces: "get the wrong idea, misinterpret, misconstrue, get the wrong impression, misread, misapprehend ...".

Of these, 'misconceptions' is probably the term that has been used most widely in the mathematics and science education literature, however, its use has declined in recent years with the recognition that individuals do not construct meaning in isolation from their socio-cultural setting. No-one sets out to invent a misconception or misconstrue their experience. Individuals make sense of their experience, shared or otherwise, on the basis of what they already know or attend to in the moment, and what is seen to be valued by the community in which the experience is situated. That is, on the basis of what represents a 'best fit' in the circumstances. In this sense, there is an argument that there are no such things as misconceptions, just *alternative conceptions* that are as good as they can be given the student's prior knowledge and their opportunities to learn. For example,

Richard (not his real name) used this method to add 19 and 27. In Year 5 at the time, Richard was known to have considerable difficulties with place-value. His reasoning here was identified by his teacher who knew that Richard was an avid AFL fan. Unable or lacking the necessary confidence to work with the base ten number system, Richard applied what he did know, rewriting the numbers in terms of football scores: 19 points is 3 goals and 1 behind, 27 points is 4 goals and 3 behinds. Adding goals, he then recorded 42, on the grounds that 7 goals is 42 points, and 4 as the sum of the behinds. Consistent with football scoring he then added the 42 points and 4 behinds to arrive at his answer of 46.

$$\begin{array}{r} 19 \\ + 27 \\ \hline 31 \\ 43 \\ \hline 424 \\ \hline 46 \end{array}$$

This is an alternative conception rather than a misconception. While the analysis identifies a starting point for teaching, this should not be the only response to Richard's work sample. The classroom practices that, over time and to various extents, have contributed to this state of affairs also need to be addressed.

Another important point to remember from a social constructivist perspective is that errors may not indicate misconceptions. For example, a Year 4 student gave the following response to the problem, 8 families shared a lottery win of \$348, how much did each family receive?

Nick (not his real name) had previously demonstrated that he could quickly and accurately solve division problems like this using *guzinta* (that is, "8 goes into 3 ... no, 8 goes into 34 ... yes, 4 times and 2 over ... 8 goes into 28 ... yes, 3 times and 4 left over"). Challenged in class over a period of three weeks to shift to a partitive approach using MAB materials, Nick demonstrated that he could physically share the materials and confidently use the related language (that is, "3 hundreds shared among 8? ... no, rename as tens, 34 tens shared among 8? ... yes, 4 tens each and 2 tens remaining, rename as ones ... 28 ones shared among 8? ... yes, 3 ones each and 4 left to share"). However, when asked to solve this word problem some time later, Nick chose to create his own algorithm based on what he knew about subtraction and renaming. In this case, reading from the top down and starting with the ones he reasoned, "8 how many 8s? ... 1 ... 4 how many 8s? Can't do, so trade a hundred, 14 tens how many 8s? ... 1 and 6 over", which he records over the 2 in the hundreds place. Realising "2 how many 8s?" is not going to work, he crosses the 6 out and rewrites the 2 and the 6 as 26 and proceeds, "26 how many 8s? ... 3 and 2 remainder" which he records. His comments

indicate his beliefs about what he sees school mathematics is about, that is, using 'sums' to get answers. When asked about his answer, Nick, said, "Oh if it was real money I wouldn't do it like that". Prompted to say how he would do it, Nick replied, "Well 8 families, \$40 each that'd be \$320, \$50 each would be \$400, I reckon it's about \$43". Nick's problem was not with division but with the values and beliefs he held about the nature and purpose of school mathematics. Asked why he did this, Nick said that, he knew his "old way of doing it would work but Mrs ... didn't like that" and he could do the new way "but that was too long."⁴

These considerations are important because they broaden the focus of attention from the individual who has an alternative conception, to the 'big ideas' and learning environments that are needed by all students to support further learning with understanding.

About the Tools

The idea behind the use of the tools is to provide teachers with a set of easy-to-use diagnostic tasks that expose critical aspects of student thinking in relation to key aspects of Number as it is this area that research has shown to be most responsible for the huge differential in student performance by the middle years. The tools also provide advice on targeted teaching responses to the 'common misunderstandings' and/or learning needs identified. They are particularly useful in identifying the learning needs of students who teachers believe are 'at risk' or likely to be at risk in relation to these important underpinnings. In some cases, this might mean using the tools from the level below the student's actual year level. On the other hand, they can also be used to obtain more accurate or in-depth information about students who teachers feel are under-achieving. In this case, it may be appropriate to use tools from the level above the student's actual year level.

A small number of 'stand-alone' tasks are provided at each Level. The tasks have been designed to be administered individually, and generally take between 5 and 10 minutes, which means that they can be used in class without withdrawing students (although sometimes this may be advisable). Before using the tools, it is suggested that teachers read the related advice so that they are aware of likely responses. As the whole point is to expose student thinking, not to assist student's to get the 'right answer', teachers are strongly advised to resist the urge to teach during these episodes and to either terminate the conversation or move on to another task as soon as a student experiences difficulty.

Wherever possible, readily available classroom materials have been used. Cards can be reproduced as required or laminated and some additional resources have been provided to support the targeted teaching suggestions. While it might seem tedious to prepare the cards, it is important to use these with individuals rather than attempt to adapt the activity to a pen and paper worksheet task that might be used with a small group or the whole class. The cards provide 'thinking space' as students have the opportunity to move them around and see them in relation to other cards. Some tasks require the cards to be manipulated or sorted, others simply serve the purpose of presenting a problem in isolation from other problems without the expectations flagged by empty spaces to provide a whole lot of working.

For each tool, the advice has been presented in a table that matches an observed response (left hand column), with an interpretation (in italics) and one or more suggested teaching responses (dot points) in the right hand column. Teachers should identify the observed response that best matches the student's response and consider how they might implement the suggested teaching response.

The tools at each Level have been chosen to address key ideas at that Level which, if not understood, will seriously undermine students' capacity to engage meaningfully with core aspects of the Number Strand in subsequent years. The key ideas addressed at each level are listed below.

LEVEL 1 – Trusting the Count, developing flexible mental objects for the numbers 0 to 10.

LEVEL 2 – Place-value, the importance of moving beyond counting by ones, the structure of the base 10 numeration system.

LEVEL 3 – Multiplicative thinking, the key to understanding rational number and developing efficient mental and written computation strategies in later years.

LEVEL 4 – Partitioning, the missing link in building common fraction and decimal knowledge and confidence.

⁴ Siemon, D. E. (1993). *The role of metacognition in children's mathematical problem solving*. An unpublished doctoral thesis, Monash University, September.

LEVEL 5 – Proportional reasoning, extending what is known about multiplication and division beyond rule-based procedures to solve problems involving fractions, decimals, per cent, ratio, rate and proportion.

LEVEL 6 – Generalising, skills and strategies to support equivalence, recognition of number properties and patterns, and the use of algebraic text without which it is impossible to engage with broader curricula expectations at this level.

It is hoped that the tools will prove a useful resource in addressing the needs of all learners but particularly those that fall behind.

TRUSTING THE COUNT

In *Number* students ... manipulate concrete and visual models to develop understanding of the fundamental concepts and objects of number [and] numeral ... They relate counting of discrete objects in sets to spatial patterns and arrangements of 1 to 20 objects with physical, visual and written representations including numerals. They apply number to establish sequence and order with respect to the elements of sets... (VELS, 2005, p.10)

Common Misunderstanding:

Many students who are able to recite the number naming sequence (ie, count orally) to 20 and beyond; recognise, read, and write number words and numerals to 10; and count and model small collections (less than 20), will guess when asked 'how many' in a particular collection or to identify which of two single digit numbers presented orally or in written form is the larger/smaller, and/or experience difficulty when counting larger collections (40 or more) accurately.

This could be due to/associated with:

- a failure to understand that counting is a strategy to determine 'how many' and/or that the last number counted says how many;
- a mismatch between the oral words and the objects counted (eg, matches objects to syllables, omits certain number names);
- a failure to organise the count to avoid counting objects already counted; and/or
- a superficial understanding of numbers 0 to 10 (ie, limited to simple counts and recognising, reading and writing number names and numerals).

By the end of Level 1 students need a deep understanding of the numbers to 10 both in terms of what they represent and how they might be reconfigured or viewed in relation to other numbers. In particular, they need to have developed flexible mental objects for each of the numbers that go beyond the recognition of number names and numerals to include rich part-part-whole knowledge based on visual imagery. This supports trusting the count in the sense that when students read, write or hear 'seven', they can imagine what that collection might look like and how it relates to other numbers. For example, they can see a *seven* in their mind's eye as *1 more than 6*, *1 less than 8*, *3 and 4*, or *5 and 2*. This is not about addition or subtraction. It is about deeply understanding what each number means.

A key indicator of the extent to which students have developed mental objects for the numbers 0 to 10 is the extent to which they can recognise collections of these numbers without counting, that is, they can subitise.

PLACE-VALUE

In *Number* students ... work with arrays of objects and base-10 materials ... to identify, order, and model the counting numbers up to 1000. By using these materials they develop understanding of patterns in the number sequence mentally, ...and [to] *count on* and *count back*. (VELS, 2005, p.13)

Common Misunderstanding:

Many students who are able to identify place-value parts (eg, they can say that there are 4 hundreds 6 tens and 8 ones in 468) and count orally to 100 and beyond, still think about or imagine 2 and/or 3 digit collections additively in terms of ones (ie, 468 is actually understood as the sum of 400 ones, 60 ones and 8 ones).

This could be due to/associated with:

- inadequate part-part-whole knowledge for the numbers 0 to 10 and/or an inability to trust the count (see Level 1 Tools);

- an inability to recognise 2, 5 and 10 as composite or countable units (often indicated by an inability to count large collections efficiently);
- little or no sense of numbers beyond 10 (eg, fourteen is 10 and 4 more); and/or
- a failure to recognise the structural basis for recording 2 digit numbers (eg, sees and reads 64 as “sixty-four”, but thinks of this as 60 and 4 without recognising the significance of the 6 as a count of tens, even though they may be able to say how many tens in the tens place)

By the end of Level 2 students need a deep understanding of the place-value pattern, *10 of these is 1 of those*, to support more efficient ways of working with 2 digit numbers and beyond. Place-value is difficult to teach and learn as it is often masked by successful performance on superficial tasks such as counting by ones on a 0-99 or 1-100 Number Chart. The structure of the base ten number system is essentially multiplicative, as it involves counts of different sized groups that are powers of 10. Unfortunately, place-value is often introduced before students have demonstrated an understanding that the numbers 2 to 10 can be used as countable units and/or before any informal work with equal groups. As a consequence, many students develop misconceptions in this area which serve to undermine their capacity to use place-value based strategies to support efficient mental and written computation and their later understanding of larger whole numbers and decimal fractions.

A key indicator of the extent to which students have developed a sound basis for place-value is the extent to which they can efficiently count large collections and confidently make, name, record, compare, order, sequence, count forwards and backwards in place-value parts, and rename 2 and 3 digit numbers in terms of their parts.

MULTIPLICATIVE THINKING

In *Number* students ... routinely use multiples to skip count and create number patterns, including multiples of 10, to explore more fully place-value and the operation of multiplication. They work on practical problems in which the complexity of computations extends to include addition and subtraction of three-digit numbers, multiplication by single digits, and division by a single-digit number.... (VELS, 2005, p.16)

Common Misunderstanding:

Although most students at this Level have some knowledge of the multiplication facts to 100 and can perform simple multiplication and division procedures correctly, many rely on rote learning and/or a naïve, *groups of* understanding for multiplication based on repeated addition (often counting equal groups by ones). With little or no access to a broader range of ideas for multiplication they find it difficult to develop efficient mental strategies, and as a consequence, tend to rely on memorised procedures for multiplying and dividing larger whole numbers and decimals.

This could be due to/associated with:

- an inability to trust the count and see numbers as countable units in their own right, that is, view 6 items as 1 six (“a six”) rather than 6 ones (see Tool 2.2);
- poorly developed or non-existent mental strategies for addition and subtraction;
- an over-reliance on physical models to solve simple multiplication problems; and/or
- a limited exposure to alternative models of multiplication.

By the end of Level 3 students need to be able to think about multiplication in a number of different ways to recognise when multiplication is required and how it relates to division, support efficient mental and written computation, and solve a wider range of problems involving equal groups, simple proportion, combinations, and rate. To do this they need to recognise the numbers 2 to 10 as countable units, count large collections more efficiently, and appreciate the advantages of representing multiplicative situations in terms of arrays and regions. That is, that arrays and regions

- more neutrally represent all aspects of the multiplicative situation, that is, the number of groups, the equal number in each group, and the product (last two not as evident in *groups of* models);
- can be used to relate the two ideas for division, *partition* (or sharing) and *quotition* (or how many groups in), to multiplication;
- support commutativity (eg, 3 fours can be rotated to show that it is the same as 4 threes) so halving the amount of learning required for the multiplication facts;
- support more efficient, generalisable mental strategies for multiplication; and
- provide a basis for moving from a count of equal groups (eg, 1 six, 2 sixes, 3 sixes, 4 sixes, ...) to a constant number of groups (eg, 6 ones, 6 twos, 6 threes, 6 fours, 6 fives ...) which supports more

efficient mental strategies (eg, 6 groups of anything is double 3 groups or 5 groups and 1 more group).

More importantly, arrays and regions support the shift from an additive *groups of* model to a *factor-factor-product* model which is needed to support fraction representation, the multiplication and division of larger whole numbers, fractions and decimals, and algebra. An awareness of the *for each* idea or Cartesian product is also needed at this Level to support work in Chance and Data (eg, problems involving combinations), measurement (including problems involving rates), and fraction representation. For example, if a diagram showing thirds is halved and halved again, there are 4 smaller parts for each third, this is not a groups of idea that corresponds to students experience.

A key indicator of the extent to which students have developed a broader range of ideas to support multiplicative thinking is the extent to which they manipulate both the size of the group and the number of groups to meet specific needs (eg, instead of committing 6 eights to memory in a meaningless or rote way, recognise that this can be thought of as 5 eights and 1 more eight, or 3 eights doubled).

PARTITIONING

In *Number* students work with the size and order of large and small numbers including negative numbers, and rational numbers in fraction and decimal form. They learn to identify natural numbers and their factors as prime, even or odd, and to use decimals, ratios and percentages to represent equivalent forms of common fractions.
(VELS, 2005, p.21)

Common Misunderstanding:

There is little doubt that a considerable proportion of Year 5 and 6 students experience difficulty with fractions, decimals and percent. A major factor contributing to this is that many students misinterpret the meaning of the denominator. Also, while students may exhibit an intuitive understanding of proportionality in terms of the *out of* idea, this is limited to familiar contexts and proper fractions (eg, 3 quarters of a pizza or the fraction of red smarties in a packet of smarties). Few students at this level see fractions as numbers which can be arrived at by partitive division (eg, 3 pizzas shared among 4) and 'live' uniquely on the number line as measures.

This could be due to/associated with:

- viewing the denominator in the same way as the numerator (ie, as a count or 'how many' number, rather than an indication of 'how much');
- a limited exposure to practical experiences that show what happens as the number of parts are increased and how fractional parts are named;
- a *groups of* only idea for multiplication and division; and
- little or no access to strategies that support the construction of appropriate fraction representations.

By the end of Level 4 students need to be able to work meaningfully with a wider range of numbers. In particular, they need to have established a meaningful basis for thinking about rational numbers in whatever form they appear (eg, proper fractions, mixed fractions, decimal fractions, and percentages). This requires the recognition that equal parts are required; that the number of parts is related to the name of the part (ie, fifths for 5 parts, sixteenths for 16 parts); that as the number of parts increases, each part becomes smaller; and that fraction representations are created by partitioning discrete or continuous quantities into equal parts (see *Partitioning* paper in Additional Resources). Understanding the relationship between fractions and partitive division is essential for fraction renaming (equivalent fractions). In particular, students need to recognise how the *region* idea for multiplication is related to fraction diagrams, for example, thirds (3 parts) by quarters (4 parts) produces twelfths (12 parts), and how increasing/decreasing the number of parts can be understood in terms of *factors*, for example, recognising that 3 parts (thirds) increased by a factor of 4 (as a result of halving and halving again) produces 12 parts (twelfths).

Key indicators of the extent to which students have developed an understanding of fractions and decimals is the extent to which they can construct their own fraction models and diagrams, and name, record, compare, order, sequence, and rename, common and decimal fractions.

PROPORTIONAL REASONING

In *Number* students learn to classify numbers encountered at earlier levels as natural numbers, integers and rational numbers. ... They comprehend and use ratio as a representation of relative size, and proportion as equivalent ratio, and learn to consider percentage as proportion relative to 100. Students develop understanding of the concept of constant rate of change in terms of constant ration between two variables. (VELS, 2005, p.27)

Common Misunderstanding:

One of the reasons many Year 7 and 8 students experience difficulty interpreting and using ratios, rates and per cent, is that they have not yet acquired a capacity for proportional reasoning. This is a complex form of reasoning that depends on many interconnected ideas and strategies developed over a long period of time. These features are amply illustrated by Lamon's (1999)⁵ description of proportional reasoning as "the ability to recognise, to explain, to think about, to make conjectures about, to graph, to transform, to compare, to make judgements about, to represent, or to symbolize relationships of two simple types ... direct ... and inverse proportion" (p.8). At its core, proportional reasoning requires a capacity to identify and describe what is being compared with what. Essentially, there are two types of proportional reasoning problems, both of which require some form of comparison. The first typically involves a comparison of two rates, eg, *Which car had a faster average speed, Car A which travelled 217 km in 1¾ hours or Car B which travelled 204 km in 1½ hours?* The second type is referred to as missing value problems. These problems typically provide 3 quantities and the fourth is missing, eg, *If a supermarket worker can unpack 24 boxes in 1 hour, how many boxes could he unpack in 10 minutes?*

Recognising what is being compared with what is not always straightforward. It can be confounded by the types of quantities used, how they are represented, and the number of variables involved. Also, not all problems in which 3 quantities are given and a fourth is missing require proportional reasoning. As Lamon notes, "There are no shortcuts available here! Thought, common sense, and experience must be used to determine whether a situation is proportional or not. You must always bring into play your knowledge about how things work in the real world" (P.225).

While this is undoubtedly true, proportional reasoning also requires a capacity to work flexibly and confidently with the quantities involved (that is, measures, rates and/or ratios expressed in terms of natural numbers, rational numbers, and/or integers), and an ability to recognise multiplicative relationships in a range of problem contexts including the idea of rational numbers as operators (eg, understanding $\frac{2}{3} \times \$24$ as $\frac{2}{3}$ of 24, or 3.5×68 as 3 and a half times 68). Neither of which, according to recent research⁶, can be assumed to be in place for all students at this level of schooling.

GENERALISING

In *Structure* students ... become familiar with the application of algebraic properties, including closure, associative, commutative, identify, inverse, and distributive, with respect to natural, integer, rational and real number systems and are able to apply them in the manipulation of mathematical expressions, formulas and equations.

In *Working Mathematically* students abstract common patterns and structural features from mathematical situations and formulate conjectures, generalisations and arguments in natural language and symbolic form (VELS, 2005, p.35).

Common Misunderstanding:

While considerably more is expected of students at this level in relation to their understanding and use of number than is expected at earlier levels (eg, see VELS, 2005, p.36), most students are able to work with rational numbers to some extent and have an emerging appreciation of the real numbers. However, this is not necessarily the case when these numbers are represented by pro-numerals or used in expressions containing pro-numerals. For many, the very power and density of algebraic text can be the feature that renders it impenetrable.

There is an extensive body of research which has examined the various difficulties students experience with algebraic text, ranging from misunderstanding of the equal sign, and assigning literal meanings to letters (eg,

⁵ Lamon, S. (1999). *Teaching Fractions and Ratios for Understanding – Essential content knowledge and instructional strategies for teachers*. Mahwah, NJ: Lawrence Erlbaum and Assoc

⁶ *Scaffolding Numeracy in the Middle Years Numeracy Research Project (2003-2006)*, an ARC Linkage project involving RMIT University, the Victorian Department of Education and Training and the Tasmanian Department of Education.

3a interpreted as 3 apples) to viewing expressions as instructions to operate, rather than as objects that can be operated on in their own right (eg, that $4x-7$ is an object that can be multiplied by any other number or pro-numeral).

While reading, interpreting, and working with algebraic text is one issue, constructing algebraic text to describe relationships is another area of difficulty for many students. A range of external representations (eg, balances, concrete materials, graphs, diagrams, or tables of values) are typically used to explore patterns and relationships in school mathematics. Referred to as *intermediate sign systems* by Filloy and Sutherland (1996)⁷, they variously serve to facilitate the construction of meaning for the conventional mathematical sign system, in this case, the “algebra code” (p.143). One of the difficulties here is that different conceptions arise from different representations and these may inhibit students’ capacity to make connections between representations, generalise, or indeed, recognise when a previously learnt representation is inappropriate. For example, while it is meaningful to interpret $5 \times \square = 20$ as ‘find the number which 5 must be multiplied by to equal 20’, this interpretation (or intermediate sign system) cannot usefully replace x in the equation, $5x + 9 = 3x$. Nor is it appropriate to expect that strategies that work for the former equation, such as ‘back-tracking’, will work with equations like the latter where the unknown appears on both sides of the equation.

The difficulties experienced in making the transition from arithmetic to algebra may be due to/associated with:

- naïve understanding of the equal sign in terms of ‘makes’ or the ‘answer is...’;
- different interpretations of letters (Booth, 1988)⁸ and/or a lack of knowledge about the conventions used to record generalised expressions (eg, that multiplication is recorded as $3a$ not $a3$ or $3 \times a$);
- limited understanding of the properties of numbers and operations (eg, multiplication only understood in terms of *groups of*, division not seen as the inverse of multiplication);
- an inadequate understanding of arithmetic and/or an over-reliance on procedural solution strategies aimed at getting numerical answers;
- little/no experience in communicating mathematical relationships in words and/or translating relationships described in words into symbolic expressions, for example, “s is 8 more than t” or the “Niger is three times as long as the Rhine” (MacGregor, 1991, pp.95-97)⁹; and
- limited access to multiplicative thinking and proportional reasoning more generally which restricts students’ capacity to recognise and describe relationships in terms of factors.

By the end of Level 6 students are expected to be able to work meaningfully with a wider range of numbers and mathematical relationships in whatever form they appear, including equations, identities, inequalities, functions and relations.

A key indicator of the extent to which students are ready to engage with these curricula expectations is their capacity to deal with equivalent forms of expressions, recognise and describe number properties and patterns, and work with the complexities of algebraic text.

⁷ Filloy, E. & Sutherland, R. (1996). Designing Curricular for Teaching and Learning Algebra. In A. Bishop, McK. Clements, C. Keitel, J. Kilpatrick, & C Laborde. (Eds.) *International Handbook of Mathematics Education*, Part 1, pp. 139-160. Kluwer: Dordrecht

⁸ Booth, L. (1988). Children’s difficulties in beginning algebra. In A. Coxford & A. Shulte (Eds.) *The Ideas of Algebra K-12, 1988 Yearbook of the National Council of Mathematics*. NCTM: Reston, VA

⁹ MacGregor, M. (1992). *Making sense of algebra: Cognitive processes influencing comprehension*. Deakin University Press: Geelong.